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Anomalous scaling and refined similarity of an active scalar in a shell model of homogeneous turbulent convection

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Anomalous scaling in the statistics of an active scalar is studied in a shell model of homogeneous turbulent convection. We extend refined similarity ideas for homogeneous and isotropic turbulence to homogeneous turbulent convection and attribute the origin of the anomalous scaling to variations of the entropy transfer rate. We verify the consequences and thus the validity of our hypothesis by showing that the conditional statistics of the active scalar and the velocity at fixed values of entropy transfer rate are not anomalous but have simple scaling with exponents given by dimensional considerations, and that the intermittency corrections are given by the scaling exponents of the moments of the entropy transfer rate.

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Since the work of Kolmogorov in 1941 (K41) [1], much effort has been devoted to the study of the possible universal statistics of fluid turbulence in the inertial range, the range of length scales that are smaller than those of energy input and larger than those directly affected by molecular dissipation. A major challenge is to understand, from first principles, the origin of anomalous scaling, which is the deviation of the velocity scaling behavior from those predicted by dimensional considerations in K41. One important idea proposed by Kolmogorov in his refined theory [2], which we refer to as Kolmogorov's refined similarity idea, attributes the origin of anomalous scaling of the velocity to the variations of local energy dissipation rate. Kraichnan [3] later pointed out that the local energy dissipation rate is not an inertial-range quantity and proposed to attribute the origin of anomalous scaling of the velocity instead to the variations of the local energy transfer rate, and we shall refer to this as Kraichnan's refined similarity idea.

Similar problems of anomalous scaling can be posed for a scalar field advected by a turbulent velocity field. A passive scalar leaves the velocity statistics intact while an active scalar couples with the velocity and influences its statistics. The nonlinear problem of anomalous scaling of active scalars, like that of velocity, remains unsolved. A common example of an active scalar is temperature in turbulent convection in which temperature variations drive the flow. Turbulent convection is often investigated experimentally in Rayleigh-Bénard convection cells heated from below and cooled on top (see, e.g., [4–6] for a review). Such confined convective flows are highly inhomogeneous with thermal and viscous boundary layers near the top and the bottom of the cell. Moreover, coherent structures, known as plumes, could affect the scaling properties [7]. For the purpose of studying anomalous scaling of an active scalar, it would thus be more desirable to study homogeneous turbulent convection and in the absence of coherent structures.

Homogeneous turbulent convection has been proposed [8] as a convective flow in a box, with periodic boundary conditions, driven by a constant temperature gradient along the vertical direction. In Boussinesq approximation [9], the equations of motion read [10]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}p + \nu\nabla^2\vec{u} + \alpha g\theta\hat{z}, \qquad (1)$$

$$\frac{\partial \theta}{\partial t} + \vec{u}\cdot\vec{\nabla}\theta = \kappa\nabla^2\theta + \beta u_z \qquad (2)$$

with $\vec{\nabla}\cdot\vec{u}=0$. Here, $\vec{u}$ is the velocity, p is the pressure divided by the density, $\theta=T-(T_0-\beta z)$ is the deviation of temperature T from a linear gradient $-\beta$, T_0, α, ν, and κ are, respectively, the mean temperature, the volume expansion coefficient, kinematic viscosity, and thermal diffusivity of the fluid, g is the acceleration due to gravity, and $\hat{z}$ is a unit vector in the vertical direction. Numerical studies [8,12] revealed that the Bolgiano length [11], given by $L_B=\epsilon^{5/4}\chi^{-3/4}(\alpha g)^{-3/2}$, where ϵ and χ are, respectively, the average energy and thermal dissipation rates, is of the order of the size of the periodic box, indicating that temperature is not active in the intermediate scales. Indeed the small-scale isotropic fluctuations were found [12] to have scaling close to that of K41.

A shell model for homogeneous turbulent convection driven by a temperature gradient has also been proposed by Brandenburg [13]. The basic idea of a shell model is to consider variables in a discretized Fourier space and construct a set of ordinary differential equations for the variables in each shell. The wave number in the nth shell is $k_n=k_0h^n$, $n=0,1,\ldots,N-1$, and h and k_0 are customarily taken to be 2 and 1, respectively. Shell models for homogeneous and isotropic turbulence have been proved to be very successful in reproducing the scaling properties observed in experiments [14]. In Brandenburg's shell model, the velocity and temperature variables u_n and θ_n are real and satisfy the evolution equations

$$\frac{du_n}{dt} + \nu k_n^2 u_n = ak_n(u_{n-1}^2 - hu_nu_{n+1}) + bk_n(u_nu_{n-1} - hu_{n+1}^2) + \alpha g\theta_n, \qquad (3)$$

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$$\frac{d\theta_n}{dt} + \kappa k_n^2 \theta_n = \tilde{a} k_n (u_{n-1}\theta_{n-1} - h u_n \theta_{n+1}) + \tilde{b} k_n (u_n \theta_{n-1} - h u_{n+1}\theta_{n+1}) + \beta u_n, \quad (4)$$

where a, b, $\tilde{a}$, and $\tilde{b}$ are positive parameters. Earlier work showed that the scaling behavior depends only on the ratio b/a [13]: close to Bolgiano-Obukhov (BO) scaling [11,16] ($u_n \sim k_n^{-3/5}$, $\theta_n \sim k_n^{-1/5}$) for b/a large and close to K41 scaling [1] ($u_n \sim k_n^{-1/3}$, $\theta_n \sim k_n^{-1/3}$) for b/a smaller than about 0.4.

In this Rapid Communication, we study anomalous scaling of an active scalar using this shell model. We show that buoyant forces are indeed significant and demonstrate explicitly the anomalous scaling behavior. Then we extend Kraichnan's refined similarity idea and attribute the origin of the anomalous scaling in homogeneous turbulent convection to variations of the entropy transfer rate. Finally we verify the consequences and thus the validity of our hypothesis against results obtained from numerical simulations of the model.

Multiplying Eq. (3) by u_n, we get the energy budget:

$$\frac{dE_n}{dt} = F_u(k_n) - F_u(k_{n+1}) - \nu k_n^2 u_n^2 + \alpha g u_n \theta_n, \quad (5)$$

where $E_n = u_n^2/2$ is the energy in the nth shell, $F_u(k_n) \equiv k_n(a u_{n-1} + b u_n) u_{n-1} u_n$ is the rate of energy transfer from $(n-1)$th to nth shell, $\nu k_n^2 u_n^2$ is the rate of energy dissipation in the nth shell due to viscosity, and $\alpha g u_n \theta_n$ is the power injected into the nth shell by the buoyant forces. It is thus reasonable to take buoyancy to be significant in the nth shell if

$$\alpha g |\langle u_n \theta_n \rangle| > \epsilon \equiv \nu \sum_n k_n^2 \langle u_n^2 \rangle, \quad (6)$$

where $\langle \cdots \rangle$ is an average over time. Note that for both K41 and BO scaling, $\alpha g \langle u_n \theta_n \rangle = \epsilon$ at $k_n = 1/L_B$.

We numerically integrate Eqs. (3) and (4) using the fourth-order Runge-Kutta method with an initial condition of $u_n = \theta_n = 0$ except for a small perturbation of θ_n in an intermediate value of n. We calculate the statistical averages when the system is in the stationary state. We find that Eq. (6) is satisfied for most of the shells when b/a is greater than a critical value of about 2 (see Fig. 1 for the case of $b/a = 100$). As in Ref. [15], a linear damping term acting on the largest scale is added to Eq. (3) to achieve stationarity when b/a is large. When b/a is smaller than the critical value, the solution is not chaotic and given approximately by the fixed-point solution of $u_n = A k_n^{-1/3}$ and $\theta_n = B k_n^{-1/3}$, which holds exactly in the limit of large N and $\nu = \kappa = \alpha g = 0$. In this case, we check that Eq. (6) is not satisfied for all shells except the $n=0$ shell (see also Fig. 1). We associate this change in the importance of buoyancy with the reported change in the scaling behavior discussed above.

As we are interested in the case of an active scalar, we focus on b/a large and study the velocity and temperature structure functions, $S_p(k_n)$ and $R_p(k_n)$:

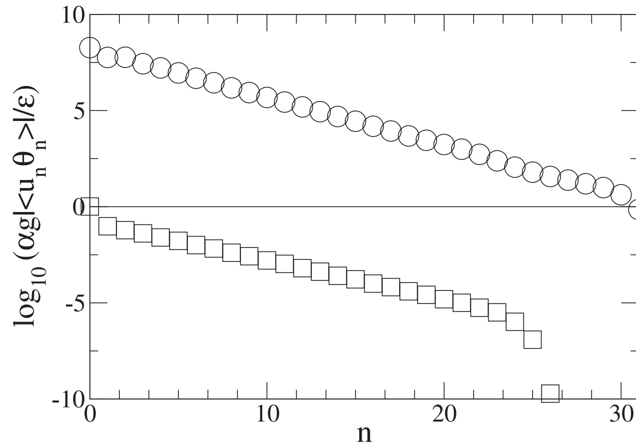

FIG. 1. The logarithm of $\alpha g |\langle u_n \theta_n \rangle| / \epsilon$ for different shells for $a=0.01$, $b=1$, $\beta=1$, $\nu=5\times10^{-17}$, $\kappa=5\times10^{-15}$, and $N=32$ (circles), and $a=10$, $b=1$, $\beta=100$, $\nu=\kappa=10^{-8}$, and $N=30$ (squares) (the datapoints for $n=28$ and 29 are not shown here as they are too small. For both cases, $\tilde{a}=\tilde{b}=1$ and $\alpha g=1$.

$$S_p(k_n) \equiv \langle |u_n|^p \rangle \sim k_n^{-\zeta_p}; \quad R_p(k_n) \equiv \langle |\theta_n|^p \rangle \sim k_n^{-\xi_p}. \quad (7)$$

The scaling exponents ζ_p and ξ_p do not depend on the values of the various parameters as long as b/a is larger than 2. The results reported below are obtained using $a=0.01$, $b=1$, $\beta=1$, $\tilde{a}=\tilde{b}=1$, $\alpha g=1$, $\nu=5\times10^{-17}$, $\kappa=5\times10^{-15}$, and $N=32$. As shown in Fig. 2, both ζ_p and ξ_p deviate, respectively, from the BO values of $3p/5$ and $p/5$, thus demonstrating anomalous scaling behavior. We study also the case where the temperature is driven by a large-scale random forcing instead of an imposed linear gradient [15]. In this case, the βu_n term in Eq. (4) is replaced by a random noise acting only in shell $n=0$. We find exactly the same scaling exponents, supporting the universality of scaling of an active scalar upon different forcing mechanisms [15,17].

It was suggested [18] that when buoyancy is dominant, the scaling behavior of velocity and temperature spectra is governed by an entropy cascade of constant entropy flux. In

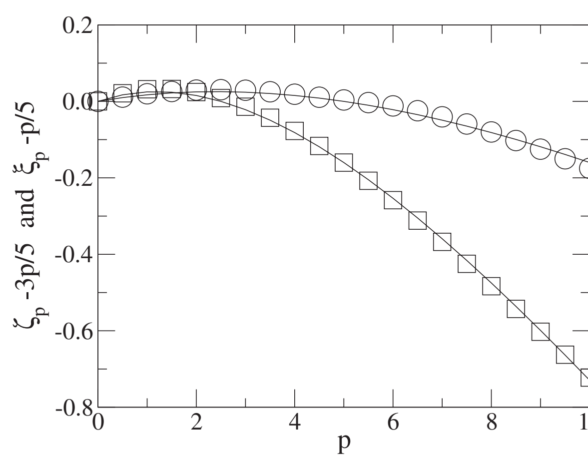

FIG. 2. Deviation of the scaling exponents from the BO values: $\zeta_p - 3p/5$ (circles) and $\xi_p - p/5$ (squares). The solid lines are the results of Eqs. (15) and (16).

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Bousinessq approximation, the entropy is proportional to the volume integral of temperature fluctuations. Entropy in the nth shell is therefore defined as $\mathcal{S}_n \equiv \theta_n^2/2$. By studying the entropy budget obtained from Eq. (4) upon multiplication by θ_n,

$$\frac{d\mathcal{S}_n}{dt} = F_\theta(k_n) - F_\theta(k_{n+1}) - \kappa k_n^2 \theta_n^2 + \beta u_n \theta_n, \qquad (8)$$

we get the rate of entropy transfer or entropy flux from $(n-1)$th to nth shell as

$$F_\theta(k_n) \equiv k_n(\tilde{a} u_{n-1} + \tilde{b} u_n)\theta_{n-1}\theta_n. \qquad (9)$$

In the stationary state and for intermediate scales where scaling is observed, both $\beta\langle u_n\theta_n\rangle$ and $\kappa k_n^2\langle\theta_n^2\rangle$ are negligible such that there is indeed a constant entropy flux with $\langle F_\theta(k_n)\rangle = \chi \equiv \kappa\Sigma_n k_n^2\langle\theta_n^2\rangle$.

We propose that when buoyancy is significant,

$$u_n = \phi_u(\alpha g)^{2/5}|F_\theta(k_n)|^{1/5}k_n^{-3/5}, \qquad (10)$$

$$\theta_n = \phi_\theta(\alpha g)^{-1/5}|F_\theta(k_n)|^{2/5}k_n^{-1/5}, \qquad (11)$$

where ϕ_u and ϕ_θ are dimensionless random variables that are independent of k_n and statistically independent of $F_\theta(k_n)$. The absolute signs are taken because the entropy flux $F_\theta(k_n)$, unlike χ, can assume both positive and negative values. Equations (10) and (11) are an extension of Kraichnan's refined similarity idea to homogeneous turbulent convection. With Eqs. (10) and (11), we attribute the anomalous scaling behavior of the active temperature and the velocity to the shell-to-shell variations of the entropy transfer rate. An immediate consequence is that the conditional velocity and temperature structure functions at a certain prescribed value x of the entropy transfer rate are given by

$$\langle |u_n|^p | F_\theta(k_n) = x\rangle = \langle\phi_u^p\rangle(\alpha g)^{2p/5}x^{p/5}k_n^{-3p/5} \sim k_n^{-\zeta_p^*}, \qquad (12)$$

$$\langle |\theta_n|^p | F_\theta(k_n) = x\rangle = \langle\phi_\theta^p\rangle(\alpha g)^{-p/5}x^{2p/5}k_n^{-p/5} \sim k_n^{-\xi_p^*}, \qquad (13)$$

and hence would have simple scaling with BO exponents of $\zeta_p^* = 3p/5$ and $\xi_p^* = p/5$, respectively. We evaluate the conditional velocity and temperature structure functions at different values of x and confirm that ζ_p^* and ξ_p^* are independent of x, and in good agreement with $3p/5$ and $p/5$, respectively, as shown in Fig. 3.

Let $\langle|F_\theta(k_n)|^p\rangle \sim k_n^{-\tau_p}$, then Eqs. (10) and (11) imply

$$\zeta_p = 3p/5 + \tau_{p/5}; \quad \xi_p = p/5 + \tau_{2p/5} \qquad (14)$$

showing that the intermittency corrections, which are the deviations of the scaling exponents from the BO values, are given by the scaling exponents of the moments of the entropy transfer rate. As the power of F_θ in R_p is twice that in S_p, this explains why the anomaly is larger for ξ_p than for ζ_p (see Fig. 2). We evaluate τ_p numerically and check Eq. (14) in Fig. 4. Good agreement is again found.

Next, we show that the intermittency corrections, as given by τ_p, can be obtained by suitably modifying the results of the scaling exponents of the moments of the local thermal

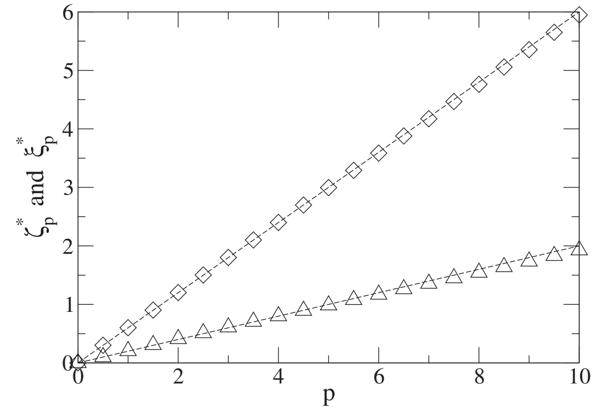

FIG. 3. The scaling exponents ζ_p^* (diamonds) and ξ_p^* (triangles) of the conditional velocity and temperature structure functions. They are in good agreement with the BO values of $3p/5$ and $p/5$ (dashed lines).

dissipation rate found in experiments [19]. In Ref. [19], the statistics of the local thermal dissipation rate, estimated by χ_τ, have been studied in the central region of turbulent Rayleigh-Bénard convection. It was found that the moments of $\chi_\tau \equiv (\langle u_c^2\rangle\tau)^{-1}\int_t^{t+\tau}\kappa(\partial T/\partial t')^2 dt'$, where $\langle u_c^2\rangle$ is the mean square velocity fluctuations at the center, satisfy a hierarchical structure of the She-Leveque form [20], and that their scaling exponents μ_p, defined by $\langle\chi_\tau^p\rangle \sim \tau^{\mu_p}$, can be well described by $\mu_p = c(1-\beta_\chi^p) - \lambda p$ with $c=1$, $\beta_\chi = 2/3$, and $\lambda = 1/3$. The parameter λ is the scaling exponent of $\lim_{p\to\infty}\langle\chi_\tau^{p+1}\rangle/\langle\chi_\tau^p\rangle$, which was estimated [19] as the ratio of the maximum thermal dissipation divided by a time t_r at the scale $r = \sqrt{\langle u_c^2\rangle}\tau$. Taking t_r as r/u_r, $b=1/3$ implies $u_r \sim r^{1/3}$, which is Kolmogorov scaling. Here, we find that the moments of the entropy transfer rate $\langle|F_\theta(k_n)|^p\rangle$ also satisfy the same hierarchical structure [21] and similarly τ_p is well approximated by $c_1(1-\gamma^p) - c_2 p$. Similarly, c_2 is the scaling exponent of $F_\theta^{(\infty)}(k_n) \equiv \lim_{p\to\infty}\langle|F_\theta(k_n)|^{p+1}\rangle/\langle|F_\theta(k_n)|^p\rangle$. Following Ref. [19], we estimate $F_\theta^{(\infty)}(k_n)$ as $\mathcal{S}_{max}u_n k_n$, where $\mathcal{S}_{max}$ is the largest possible entropy. Since we observe BO-like scaling in the present case, it is more appropriate to

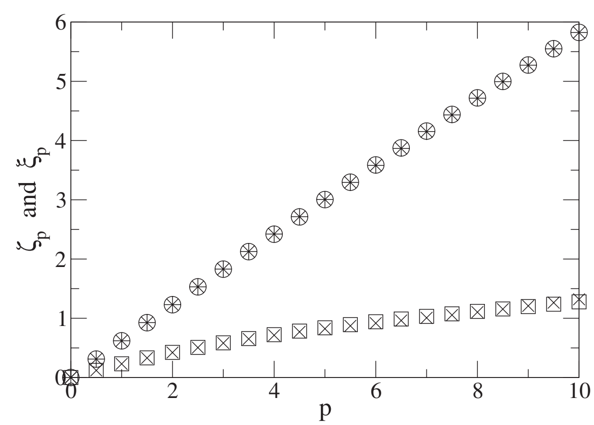

FIG. 4. Comparison of ζ_p (circles) and ξ_p (squares) with the theoretical predictions of $3p/5 + \tau_{p/5}$ (stars) and $p/5 + \tau_{2p/5}$ (crosses) using the numerical results of τ_p.

estimate $u_n \sim k_n^{-3/5}$. As a result, we get $c_2=2/5$. Then as $\langle |F_\theta(k_n)| \rangle \sim \langle F_\theta(k_n) \rangle = \chi$ is independent of k_n, we have $\tau_1=0$ implying $c_1(1-\gamma)=c_2$. If we keep $c_1=1$ as $c=1$ for χ_τ, then we get $\gamma=3/5$. Putting these results together and using Eq. (14), we find

$$\zeta_p - 3p/5 = 1 - (3/5)^{p/5} - 2p/25, \tag{15}$$

$$\xi_p - p/5 = 1 - (3/5)^{2p/5} - 4p/25. \tag{16}$$

Interestingly, as shown in Fig. 2, Eqs. (15) and (16) indeed describe the measured intermittency corrections well.

We have focused on understanding the origin of anomalous scaling of an active scalar in homogenous turbulent convection using a shell model. We have extended Kraichnan's refined similarity idea and attributed the anomalous scaling to the variations in the entropy transfer rate. We have verified our hypothesis by showing explicitly that the conditional velocity and temperature structure functions at fixed values of the entropy transfer rate have simple scaling exponents of the BO values, and the intermittency corrections are given by the scaling exponents of the entropy transfer rate. Furthermore, by modifying earlier results obtained for the statistics of the local thermal dissipation rate in turbulent Rayleigh-Bénard convection [19], we have obtained the scaling exponents τ_p of the moments of the entropy transfer rate and thus Eqs. (15) and (16) for the intermittency corrections $\zeta_p - 3p/5$ and $\xi_p - p/5$. These results are found to be in good agreement with those obtained in the numerical simulations of the shell model.

We should note that the scaling behavior of homogeneous turbulent convection might not be the same as that in the central region of confined turbulent convection as coherent structures present in the latter case could affect the scaling properties [7]. Indeed direct numerical simulations [22] and analyses of experimental data [23] indicated that the scaling behavior of the central region of confined turbulent convection is not well described by BO scaling plus intermittency corrections. On the other hand, there is evidence [24] of the validity of the extension of Kolmogorov's refined similarity idea in terms of the local thermal dissipation rate. It would be interesting to further investigate this issue.

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